THE UPSCATTER FRACTION AND ALL THAT

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The Resurgence of the Atmospheric Radiation Subject since Manabe-Wetherald (1967)

A Wiscombe-fest:

The many joys of clouds, droplets, radiative transfer, and so on

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GLOBAL AVERAGE DIRECT FORCING BY AEROSOLS

Dependence on ***upscatter fraction***

$$F_{\text{aer}} = \frac{1}{2} J_{\text{S}} \, T^2 \, (1 - f_{\text{cld}}) \, (1 - R_{\text{sfc}})^2 \, \tau_{\text{aer}} \, \overline{\beta}$$

F_{aer}, aerosol scattering forcing

J_{S}, solar constant

T, atmospheric transmittance above aerosol

f_{cld}, cloud fraction

R_{sfc}, surface reflectance

τ_{aer}, aerosol optical depth

$\overline{\beta}$, average upscatter fraction

Upscatter fraction $\boldsymbol{\beta}$ is fraction of scattered light scattered into upward hemisphere.

WHAT IS AEROSOL FORCING AS FUNCTION OF SOLAR ZENITH ANGLE?

$$F_{\text{aer}}(\theta_0) = J_{\text{S}}\; T^2\; (1 - R_{\text{sfc}})^2\; \tau_{\text{aer}}\; \beta(\theta_0)$$

Need to know ***Upscatter fraction*** $\boldsymbol{\beta}$ as a function of solar zenith angle θ_0.

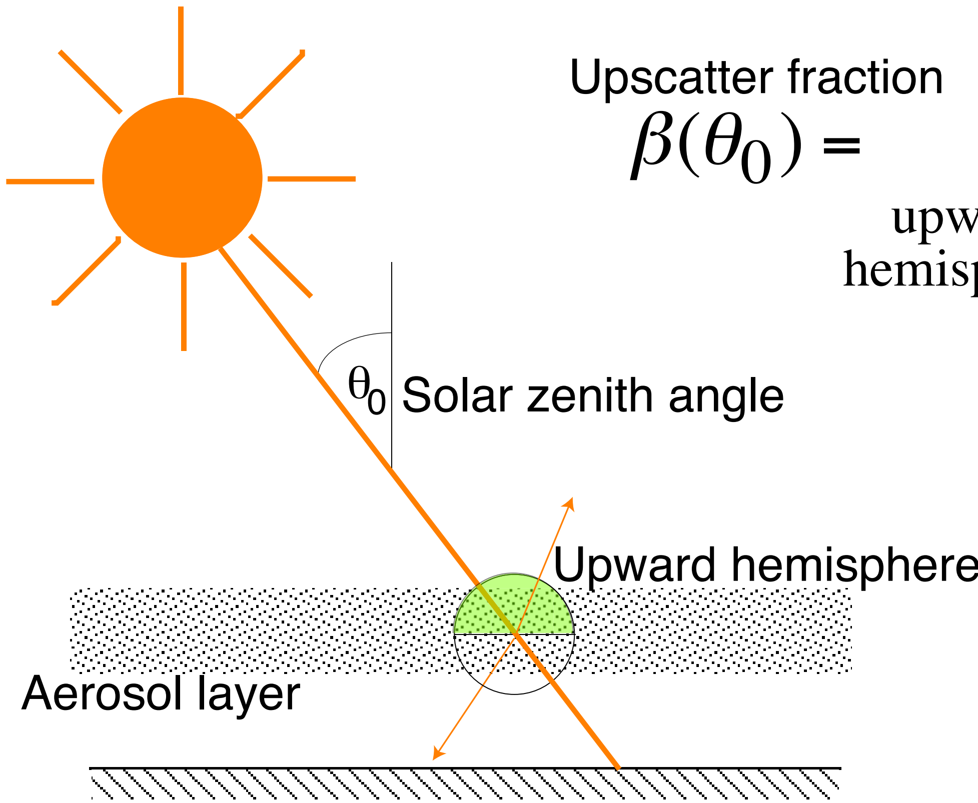
UPSCATTER FRACTION

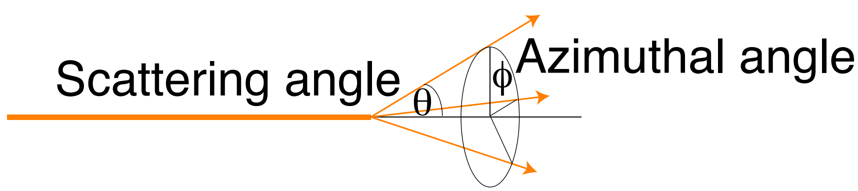
Fraction of scattered light scattered into upward hemisphere

Upscatter fraction

Phase function

$$\beta(\theta_0) = \int_{\substack{\text{upward}\\ \text{hemisphere}}} P(\theta,\phi)\,d\Omega \Big/ \int_{4\pi} P(\theta,\phi)\,d\Omega$$





Depends on solar zenith angle θ_0.

Depends also on scattering angle θ and azimuthal angle ϕ.

To get $\beta(\theta_0)$ need to integrate over θ and ϕ.

JOURNAL OF THE ATMOSPHERIC SCIENCES

Upscatter

The ~~Backscattered~~ Fraction in Two-Stream Approximations

W. J. Wiscombe and G. W. Grams

December 1976

$$\beta(\mu) = (2\pi)^{-1} \int_{\pi/2-\theta}^{\pi/2+\theta} \cos^{-1}(\operatorname{ctn}\theta \operatorname{ctn}\theta') P(\cos\theta') \sin\theta' d\theta' + \frac{1}{2} \int_{\pi/2+\theta}^{\pi} P(\cos\theta') \sin\theta' d\theta'.$$

194 Citations

WISCOMBE-GRAMS DERIVATION OF UPSCATTER FRACTION FORMULA

$$\beta(\mu_0) \equiv \frac{1}{2} \int_0^1 \bar{P}(-\mu', \mu_0)\, d\mu',$$

$$\bar{P}(\mu,\mu') \equiv \frac{1}{\pi} \int_0^{\pi} P(\mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}} \cos\phi)\, d\phi, \quad (2)$$

If we expand the phase function in Legendre polynomials P_n, viz.

$$P(\mu) = \sum_{n=0}^{\infty} \omega_n P_n(\mu),$$

then the addition theorem for spherical harmonics and Eq. (2) lead to

$$\bar{P}(\mu,\mu') = \sum_{n=0}^{\infty} \omega_n P_n(\mu) P_n(\mu').$$

the expansion coefficients ω_n are given by

$$\omega_n=\tfrac{1}{2}(2n+1)\int_{-1}^{1} P(\mu)P_n(\mu)d\mu, \qquad (13)$$

If we now insert the definition (13) of the expansion coefficient ω_n, and again switch sum and integral, we arrive at

$$\beta(\mu)=\frac{1}{2}-\frac{1}{4}\int_{-1}^{1} A(\mu,\mu')P(\mu')d\mu', \qquad (18)$$

where

$$A(\mu,\mu')\equiv\frac{1}{\pi^{\frac{1}{2}}}\sum_{m=0}^{\infty}(-1)^m(2m+\tfrac{3}{2}) \times\frac{\Gamma(m+\frac{1}{2})}{\Gamma(m+2)}P_{2m+1}(\mu)P_{2m+1}(\mu'). \qquad (19)$$

When confronted with a product of Legendre polynomials with different arguments, such as in the last sum, it is often helpful to use an addition theorem to replace the product by a single Legendre polynomial with a more complex argument. The following such relation proves useful in the present case:

$$P_n(\mu)P_n(\mu') = \pi^{-1}\int_0^\pi P_n[\mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}}\cos\phi]d\phi.$$

Putting this into Eq. (19) leads to

$$A(\mu,\mu') = \pi^{-1}\int_0^{\pi} Q[\mu\mu' + (1-\mu^2)^{\frac{1}{2}}(1-\mu'^2)^{\frac{1}{2}} \cos\phi]d\phi, \quad (20)$$

where $Q(\mu) \equiv \dfrac{1}{\pi^{\frac{1}{2}}} \sum_{m=0}^{\infty} (-1)^m (2m+\frac{3}{2}) \dfrac{\Gamma(m+\frac{1}{2})}{\Gamma(m+2)} P_{2m+1}(\mu)$.

From Mangulis (1965, p. 125), $Q(\mu)$ is simply a step function,

$$Q(\mu) = \begin{cases} -1, & -1 \leqslant \mu \leqslant 0 \\ +1, & 0 < \mu \leqslant 1. \end{cases}$$

Thus the integral over Q in Eq. (20) is trivial.

so A is either $+1$ or -1, independent of θ and θ'. Summing up, we have

$$A(\mu,\mu')=\begin{cases} +1, & 0\leqslant\theta'\leqslant\dfrac{\pi}{2}-\theta \\[2ex] 1-2\pi^{-1}\cos^{-1}(\operatorname{ctn}\theta\operatorname{ctn}\theta'), & \dfrac{\pi}{2}-\theta<\theta'<\dfrac{\pi}{2}+\theta \\[2ex] -1, & \dfrac{\pi}{2}+\theta\leqslant\theta'\leqslant\pi \end{cases}$$

From this equation and Eq. (3a), Eq. (18) becomes

$$\beta(\mu) = (2\pi)^{-1} \int_{\pi/2-\theta}^{\pi/2+\theta} \cos^{-1}(\operatorname{ctn}\theta \operatorname{ctn}\theta') P(\cos\theta') \sin\theta' d\theta'$$

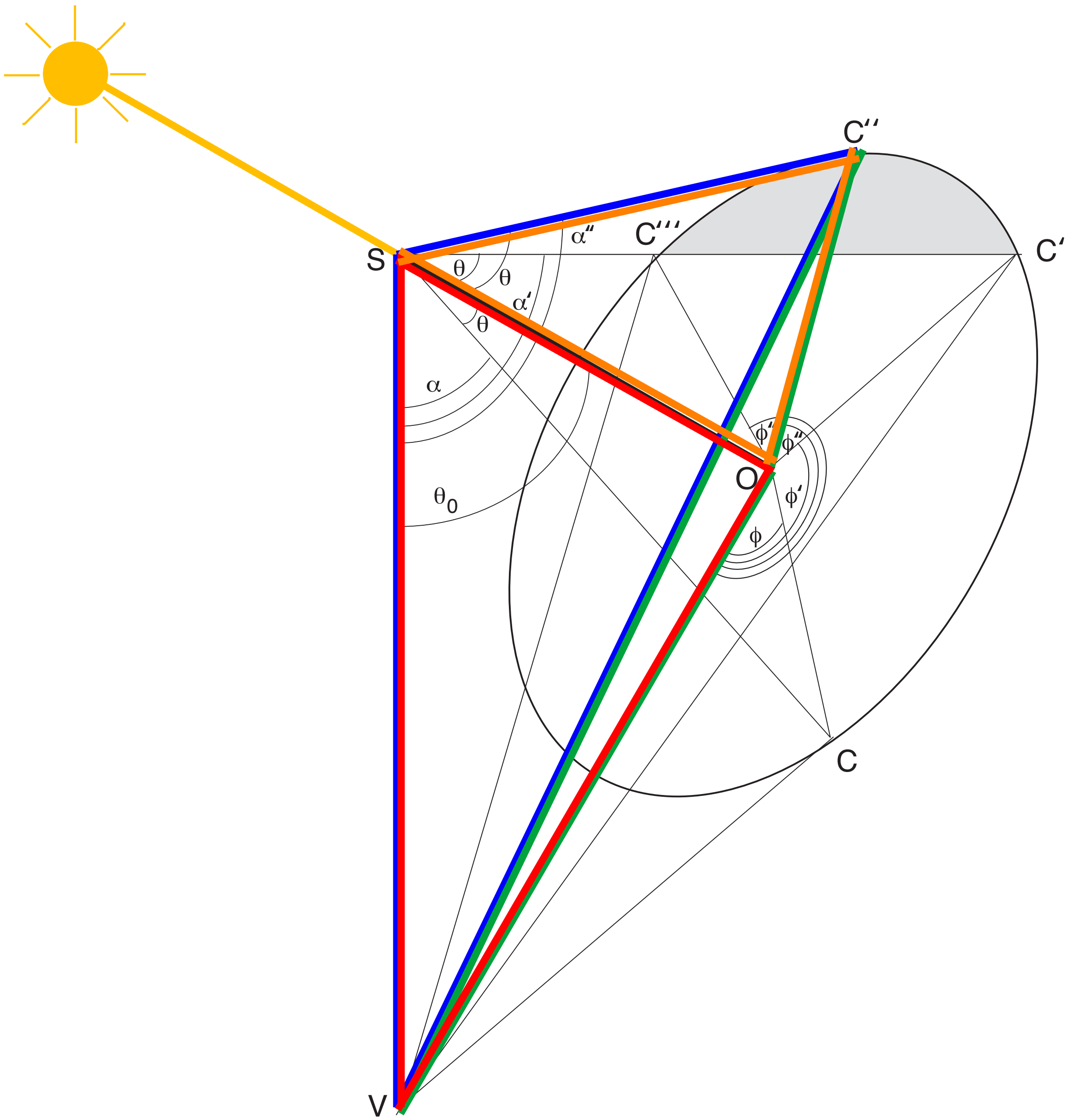
$$+ \frac{1}{2} \int_{\pi/2+\theta}^{\pi} P(\cos\theta') \sin\theta' d\theta'. \quad (22)$$

This gives $\beta(\mu)$ as single integrals over the phase function.

While at first we thought Eq. (22) was new, not having seen it in any of the more recent literature, the historical survey for Table 1 revealed that Dietzius (1922) gave Eq. (22) to within a constant factor. He gave no derivation, but J. A. Coakley (private communication) has shown that Eq. (22) follows from purely geometrical reasoning about the backscattering, which is probably how Dietzius got it. Nevertheless, our purely analytic derivation is new, and indeed becomes, to the best of our knowledge, the only extant derivation of Eq. (22) in the literature.

Wiscombe and Grams, JAS, 1976

GEOMETRICAL DERIVATION OF UPSCATTER FRACTION FORMULA



S is scattering center

θ_0 is solar zenith angle

θ is scattering angle

ϕ is azimuthal angle

α is zenith angle of scattered light

Apply law of cosines (twice)

Set inequalities using Pythagoras

An expression for this fraction can be obtained entirely from geometrical considerations. Application of the law of cosines to the triangle *OVC* gives:

$$VC^2 = OV^2 + OC^2 - 2OV \cdot OC \cos\phi \tag{4}$$

Similarly the law of cosines applied to the triangle *SVC* gives:

$$VC^2 = SV^2 + SC^2 - 2SV \cdot SC \cos\alpha \tag{5}$$

The requirement for radiation to be scattered in the upward direction is that the angle of the scattered radiation relative to the downward vertical $\alpha > \pi/2$ or $\cos\alpha \leq 0$, whence

$$VC^2 \geq SV^2 + SC^2 \tag{6}$$

and hence

$$OV^2 + OC^2 - 2OV \cdot OC \cos\phi \geq SV^2 + SC^2 \tag{7}$$

or

$$-2OV \cdot OC \cos\phi \geq SV^2 - OV^2 + SC^2 - OC^2 \tag{8}$$

Note that triangles SOV and SOC are both right triangles. Hence

$$-2OV \cdot OC\cos\phi \geq SO^2 + SO^2 \tag{9}$$

and

$$-\cos\phi \geq \frac{SO^2}{OV \cdot OC} = \cot\theta_0 \cot\theta \tag{10}$$

Hence the requirement for upscatter is that

$$-\cos\phi \geq \cot\theta_0 \cot\theta \tag{11}$$

By symmetry we may restrict ourselves to $0 \leq \phi \leq \pi$ for which $-\cos\phi$ monotonically increases with increasing ϕ so the range of ϕ for which the scattering is in the upward direction is

$$\phi \geq \cos^{-1}(\cot\theta_0 \cot\theta)\,. \tag{12}$$

The corresponding fraction of the azimuthal scattering that is in the upward direction is

$$\beta(\theta_0,\theta) = \cos^{-1}(\cot\theta_0 \cot\theta)\,/\,\pi \tag{3}$$

For a given solar zenith angle θ_0 the fraction of the scattered radiation that is scattered in the upward direction is given by the average of $\beta(\theta_0,\theta)$ over the phase function. This average consists of two terms, the integration over the range $[\theta_{min}, \theta_{max}]$ for which only a fraction of the scattered radiation is in the upward direction and the integration over the range $[\theta_{max}, \pi]$ for which the scattering is entirely in the upward direction.

$$\beta(\theta_0) = \frac{2\pi \int_{\pi/2-\theta_0}^{\pi/2+\theta_0} P(\theta)\,[\cos^{-1}(\cot\theta_0 \cot\theta)/\pi]\,\sin\theta\,d\theta + 2\pi \int_{\pi/2+\theta_0}^{\pi} P(\theta)\sin\theta\,d\theta}{2\pi \int_0^{\pi} P(\theta)\sin\theta\,d\theta} ;(13)$$

The factor 2π before each integral results from the integration over the azimuthal angle. For the phase function normalized to 4π, i.e., $\int P d\Omega = 4\pi$, the integral in the denominator is 2, and hence we obtain the result given by WG76,

$$\beta(\theta_0) = \frac{1}{2} \int_{\pi/2-\theta_0}^{\pi/2+\theta_0} P(\theta)\,[\cos^{-1}(\cot\theta_0 \cot\theta)/\pi]\,\sin\theta\,d\theta + \frac{1}{2} \int_{\pi/2+\theta_0}^{\pi} P(\theta)\sin\theta\,d\theta\,. \quad (2)$$

GEOMETRICAL DERIVATION OF UPSCATTER FRACTION FORMULA

Upscatter fraction for light scattering by atmospheric aerosols

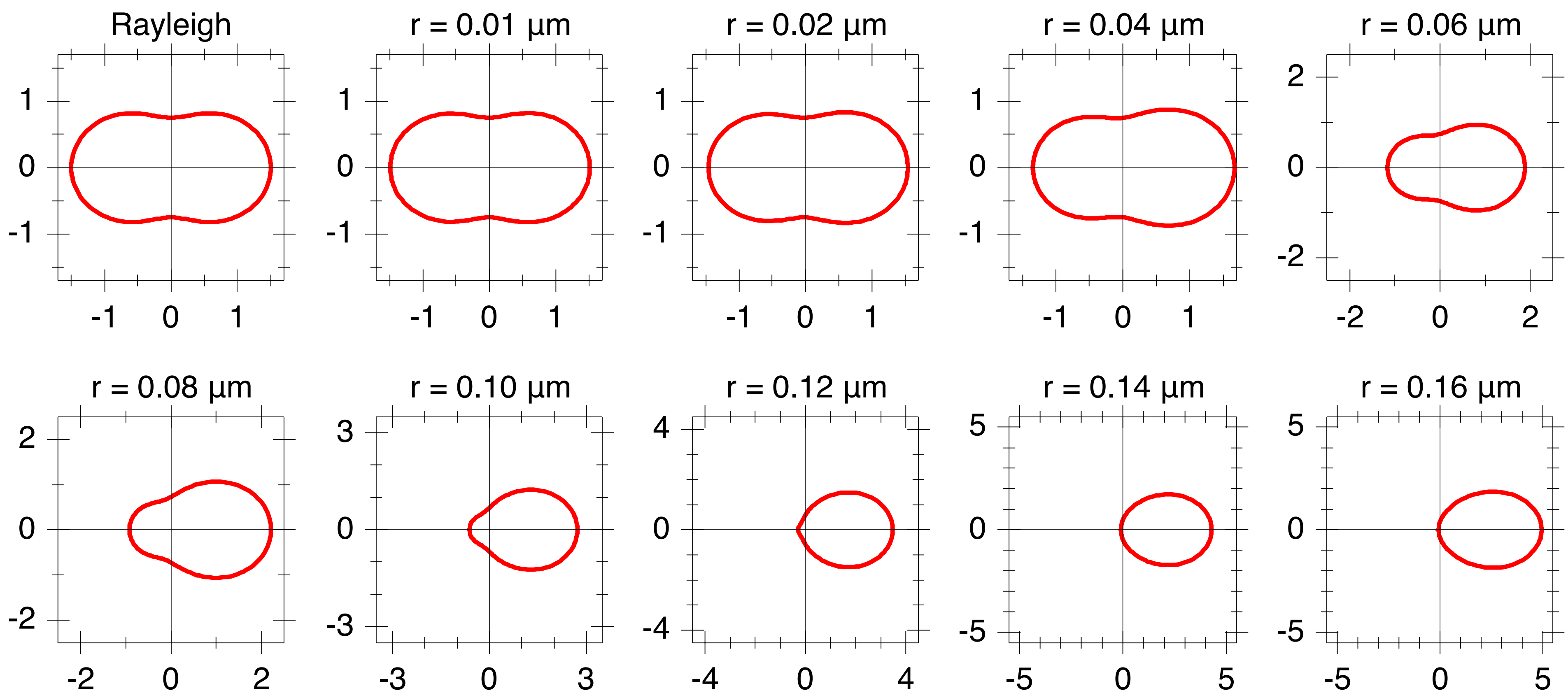
Stephen E. Schwartz, Warren J. Wiscombe, and James A. Coakley, Jr.

Draft, July, 2007

ABSTRACT. The upscatter fraction, the fraction of solar radiation scattered by an atmospheric aerosol that is scattered into the upward direction in single scattering, is of importance in consideration of radiative effects of atmospheric aerosols. While a simple expression for this upscatter fraction has been presented previously, and while it has been previously stated that this expression may be readily obtained on geometric grounds, it does not appear that the geometric derivation of this expression has previously been presented. Here we present such a derivation in the hope that it will lead to enhanced physical insight.

LIGHT SCATTERING BY AEROSOL PARTICLES

Dependence of angular distribution of scattering (phase function) on particle size

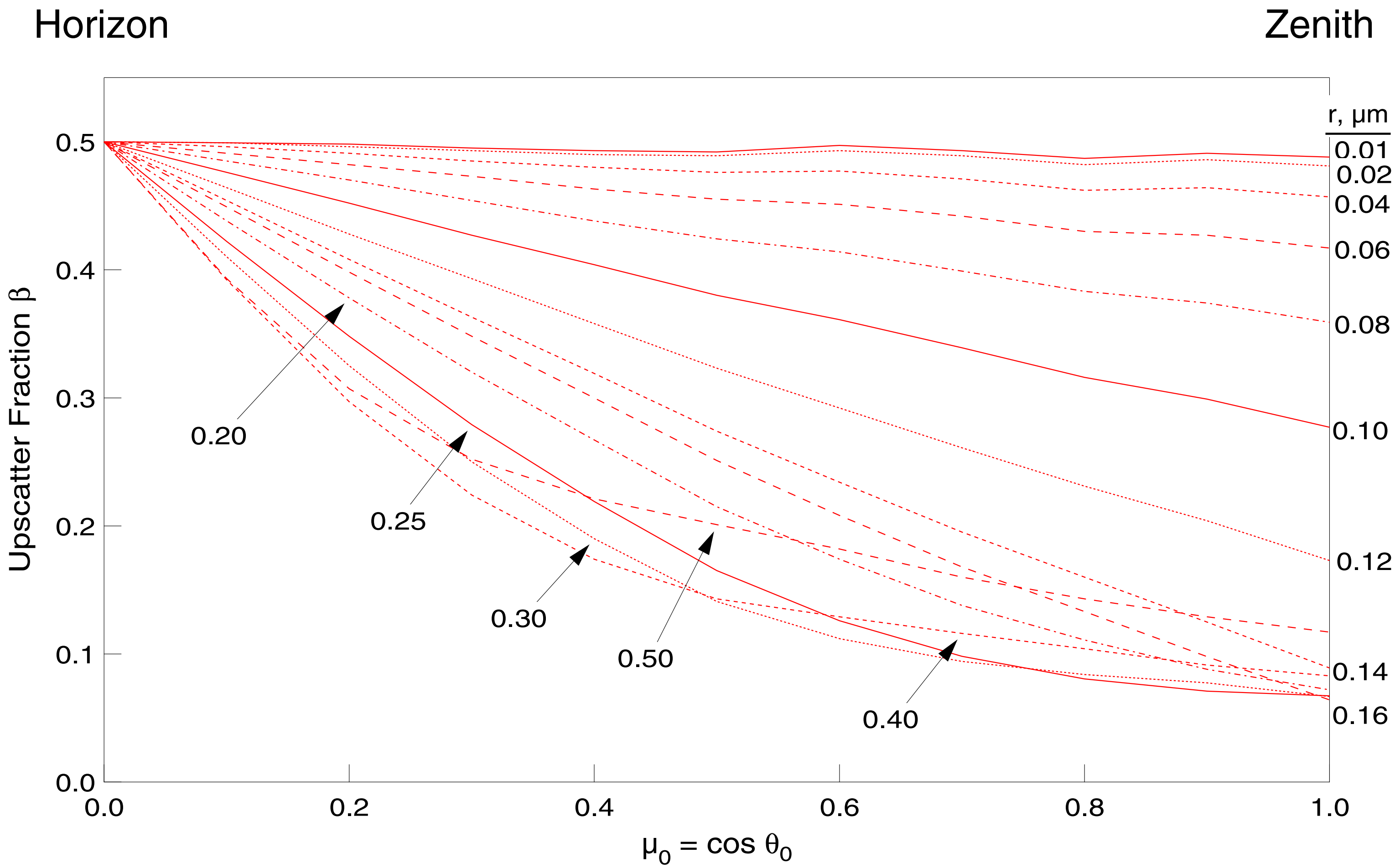


Calculations by R. Wagener, BNL

Larger particles scatter increasingly in the forward direction.

UPSCATTER FRACTION

Dependence on solar zenith angle and particle radius



$\lambda = 0.55\ \mu m$. $n = 1.40 - 0i$.

For sun at horizon $\beta = 0.5$ (by symmetry).

For small particles, $r \ll \lambda$, upscatter fraction approaches that for Rayleigh scattering (0.5).

PUZZLERS

1. For ***constant upscatter fraction***, because solar irradiance varies as cosine of solar zenith angle (greatest for overhead sun), you would think forcing by aerosol scattering would exhibit similar dependence, but ***you would be very wrong***. Why?

2. Aerosol forcing is greatest at solar zenith angle about 75° (sun 15° above horizon). Why?